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Potential Flow past a Sinusoidal Wall by Direct Variation

NORMAN L. WHITLEY

Department of Mechanical Engineering, University of New Orleans, New Orleans, Louisiana 70148

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The problem of semi-infinite potential flow past a sinusoidal wall is solved by direct variation. The direct variation proceeds from the theorem of Kelvin that states that potential flows in simply connected domains possess a minimum kinetic energy. The difficulties due to the semi-infinite domain are discussed and resolved. Trial functions which are natural solutions of the field equation and the far-field boundary condition, and which satisfy the periodicity of the flow are used. The resulting system of linear equations becomes ill-conditioned as the wall height and number of trial solutions are increased, but accurate evaluations of streamlines and field velocities for wall heights up to 0.9 are found by using established numerical methods.

1. INTRODUCTION

The problem of semi-infinite potential flow past a sinusoidal wall is familiar to most every applied mathematician. It is a model problem of the regular perturbation type and it is in this context that many students of applied mathematics first encounter it. Until recently the most thorough treatment of this problem had been that of Kaplan [1]. He had chosen the problem so as to discuss two common iteration techniques used at the time. He pursued the regular perturbation series in infinitesimal wall height by calculating by hand several higher-order corrections. He also reduced the problem to solving a nonlinear integral equation similar to that of Theodorsen and Garrick [2].

This problem has also been solved using the more modern approach of numerical methods, in particular, the finite element method. Baker and Manhardt [3] have published solutions for some wall heights and Lawkins [4] has completed a study of the accuracy and convergence of the finite element technique as it has been applied to this problem. Some of this material is readily available in the book by Baker [5]. The solution that appears there is for a very small wall height and the authors judge the accuracy of their solution by comparing it with the analytic solution for the infinitesimal wall, i.e., a solution that is linear in wall height. They find that by using 64 biquadratic elements they get excellent convergence and accuracy for a wall height wavelength ratio of 0.025.

The author [6] recently published a solution using the regular perturbation

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technique where the series had been carried out to fiftieth order by computer. Although this method produced very accurate answers for velocities for wall heights up to and beyond any reasonable physical limit, it is not a convenient method by which to pursue many different field velocities or streamlines for it requires the storage and manipulation of hundreds of coefficients. For example, a solution with 50 powers of the nondimensional wall height requires 600 coefficients. The manipulation of these coefficients to produce field velocities requires care, and the series solution does not provide a straight forward way to produce streamlines. Because of these difficulties an alternative approach involving fewer computed quantities was attempted. The method chosen was that of direct variation.

In the case of potential flows there are variational principles dating back to 1849, when Lord Kelvin stated, "The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal motion of the boundary." This statement and its proof can be readily found in the popular classic by Lamb [7]. Much later variational principles for compressible potential flow and free-surface gravity potential were found by Bateman [8] and Luke [9], respectively. In light of the details of these latter principles it can be said that it is pressure being extremized in all of these potential flows. In the case of the incompressible flow without a free surface the pressure and kinetic energy sum to a constant by Bernoulli's equation, and thus Kelvin's statement is true. Although these variational principles have existed for over a hundred years they are just recently enjoying widespread use as the backbone of many numerical methods.

In the present problem of flow past a sinusoidal wall there is a slight difficulty in implementing Kelvin's principle. Here, even though the domain is simply connected, it is infinite in extent. The kinetic energy is thus also infinite and the variational procedure is asked to minimize something that is fixed as infinite. This difficulty may be dealt with in two steps. First, reconize that the flow is periodic in the direction along the wall and therefore to minimize over one period is to minimize over the whole range. Second, seek to minimize the "perturbation" kinetic energy, that being the total kinetic energy minus the kinetic energy of the free stream.

2. Solution by Direct Variation

The nondimensional mathematical problem to be solved is this: Let the x component of the velocity u and the y component of the velocity v be given as:

$$u = 1 + \Phi_x, \tag{1}$$

$$v = \Phi_y, \tag{2}$$

where Φ is the perturbation velocity potential. Given that the wall is $y = \varepsilon \cos x$ and that the velocity tends to a uniform stream as $y \to \infty$, one must solve this problem;

$$\nabla^2\Phi = 0, \qquad -\infty < x < \infty, \cos x < y < \infty, \tag{3}$$

$$\Phi_y + \Phi_x \varepsilon \sin x + \varepsilon \sin x = 0, \qquad y = \varepsilon \cos x, \ -\infty \leqq x \leqq \infty, \tag{4}$$

$$\Phi_y \to 0 \quad \text{as} \quad y \to \infty, \qquad -\infty \leqq x \leqq \infty. \tag{5}$$

In view of the previous comments I proceed by defining the quantity to be minimized J as

$$J = \int_{-\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} [\tfrac{1}{2}(u^2 + v^2) - \tfrac{1}{2}]\, dy\, dx. \tag{6}$$

A derivation of the governing equations from this quantity may be found in Appendix A.

Here a direct variational or Rayleigh–Ritz method will be used to approximate the solution. A finite sum of trial functions will be substituted into the expression for J, the integration will be performed, and then the variation will be taken. As trial functions exact solutions of the field equation and the far-field boundary condition will be used. In specific let:

$$\Phi = \sum_{n=1}^{N} A_n e^{-ny} \sin nx. \tag{7}$$

Assuming that the series for Φ is adequately convergent such that the summations may be moved outside the integration the expression for J is

$$J = \sum_{n=1}^{N} \sum_{i=1}^{N} A_n A_i \frac{ni}{n+i} (-1)^{n-i} \pi I_{n-1}((n+i)\varepsilon) - \sum_{n=1}^{N} A_n (-1)^{n-1} 2\pi I_n(n\varepsilon), \tag{8}$$

where $I_k(t)$ is the modified Bessel function of the first kind of order k. The recurrence formula for $I_k(t)$,

$$I_{k-1}(t) - I_{k+1}(t) = \frac{2k}{t} I_k(t) \tag{9}$$

has been used in deriving Eq. (8).

The first variation of J may be accomplished by setting all derivatives with respect to A_n equal to zero, thus

$$\frac{\partial J}{\partial A_n} = 0, \qquad n = 1, 2, \ldots, N \tag{10}$$

and these being zero imply

$$\sum_{i=1}^{N} A_i \frac{i}{n+i} (-1)^i I_{n-i}((n+i)\varepsilon) = -I_n(n\varepsilon)/n, \qquad n = 1, 2, ..., N, \tag{11}$$

where the relationship

$$I_{-n}(t) = I_n(t) \tag{12}$$

has been used.

The direct variational method has reduced the problem to solving N linear equations with N unknowns. The set of Eqs. (11) were modified to control the size of the matrix coefficients. Since the modified Bessel functions $I_n(t)$ grow exponentially with argument, the nth equation was divided by $e^{n\varepsilon}$ and a new quantity was defined by

$$A_n = (-1)^n B_n e^{n\varepsilon}. \tag{13}$$

3. Computer Implementation and Results

The solution of this linear system of equations requires the accurate evaluation of the products $I_n(t) e^{-t}$. This was accomplished by the method of Lentz [10]. This method is based on the theory of continued fractions and is both very accurate and computationally efficient. The method requires an externally accurate evaluation of $I_0(t) e^{-t}$. This was acquired by using the ascending series and asymptotic expansions for this quantity which can be found in Abramowitz and Stegun [11]. The number of terms and the regions of use were chosen such that $I_0(t) e^{-t}$ could be evaluated with an error of the order of 10^{-12}.

The linear system was solved on a DEC8600 machine using double precision Gaussian elimination with partial pivoting and iterative quadruple precision refinement. The program used also produced an estimate of the number of accurate digits in the solution. By this estimate the ill-conditioning of the system was easily discerned. As ε and N were increased the system became ill-conditioned. That is, the larger ε the more rapidly the symptoms of ill-conditioning appeared with increasing N. Numerous attempts were made to improve the condition number of the matrix and although some were succesful none actually resulted in solutions that were significantly more accurate.

For $\varepsilon = 0.1$, N was chosen to be 10 and the number of accurate digits was estimated to be 15. This choice resulted in a smallest coefficient being less than 10^{-11}. For $\varepsilon = 0.3$, N was chosen to be 15 and still the number of accurate digits was estimated to be 15. For $\varepsilon = 0.6$ and 0.9, N was chosen to be 22 and 16 respectively. In both of these cases, the criteria for choosing N was that it should be the largest value for which the computer program estimated the number of accurate digits in the solution to be no less than 10. Table I contains these coefficients.

TABLE I

Coefficients for the Streamfunction

n	$\varepsilon=0.1$	$\varepsilon=0.3$	$\varepsilon=0.6$	$\varepsilon=0.9$
	A_n	A_n	A_n	A_n
0	0.4987557419 − 02	0.4402733222 − 01	0.1660410017 00	0.3443170190 00
1	0.1001254661 00	0.3034833934 00	0.6300460933 00	0.1010285845 01
2	0.5020862022 − 02	0.4670865345 − 01	0.2083987735 00	0.5564448302 00
3	0.3775322039 − 03	0.1075247732 − 01	0.1024639890 00	0.4479180510 00
4	0.3364330364 − 04	0.2932606066 − 02	0.5963019949 − 01	0.4137623025 00
5	0.3293746312 − 05	0.8786212955 − 03	0.3810133459 − 01	0.3954868381 00
6	0.3423528843 − 06	0.2794614980 − 03	0.2581523929 − 01	0.3654215687 00
7	0.3709054364 − 07	0.9264817830 − 04	0.1815727456 − 01	0.3101727747 00
8	0.4141635512 − 08	0.3166211899 − 04	0.1300427230 − 01	0.2326034633 00
9	0.4703160863 − 09	0.1107079957 − 04	0.9238796957 − 02	0.1494163173 00
10	0.4845740643 − 10	0.3933096717 − 05	0.6241350866 − 02	0.8003611217 − 01
11		0.1401907775 − 05	0.3722319963 − 02	0.3480451496 − 01
12		0.4852228549 − 06	0.1649408223 − 02	0.1190923985 − 01
13		0.1510676928 − 06	0.1415210118 − 03	0.3074840630 − 02
14		0.3628518207 − 07	−0.6967151598 − 03	0.5616464720 − 03
15		0.4807980572 − 08	−0.9169574347 − 03	0.6456650628 − 04
16			−0.7403585728 − 03	0.3507968934 − 05
17			−0.4384895937 − 03	
18			−0.1958719479 − 03	
19			−0.6485005824 − 04	
20			−0.1510059439 − 04	
21			−0.2214152129 − 05	
22			−0.1540649658 − 06	

Having the coefficients of Φ one naturally has the coefficients of the stream function Ψ, where

$$\Psi = y - A_0 - \sum_{n=1}^{N} A_n e^{-ny} \cos nx \tag{14}$$

and A_0 is the value of Ψ on the wall. The value of A_0 was determined by taking an average of Ψ at 32 evenly spaced points on the wall over one period. It is therefore straightforward to compute the streamlines of the flow using some simple root solver. In this case a modified regula falsi method, often called the Illinois algorithm [12], was used.

Figures 1–4 are plots of streamlines for wall heights $\varepsilon = 0.1$, 0.3, 0.6, and 0.9, respectively. As can be seen these truncated series produce accurate portrayals of the wall shape. Even for $\varepsilon = 0.9$ the maximum error from the true wall is estimated to be less than 3×10^{-3} and the series are much more accurate in the other cases shown here.

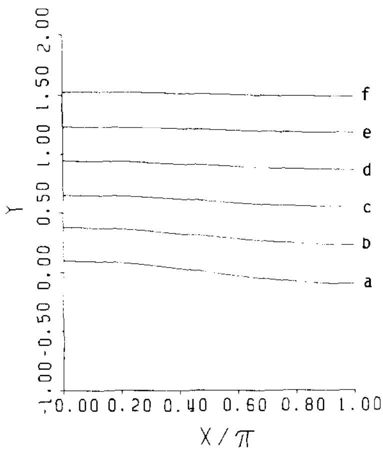

FIG. 1. Streamlines of the flow for $\varepsilon = 0.1$. The contours shown are for Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

The field velocities may also be simply computed through derivatives of Φ or Ψ. However, since the velocities depend on the derivative of Φ or Ψ, their series are more slowly convergent than the original ones. In the case of a truncated series less accurate answers for the velocities can be the result. Here, for small ε there is no problem, but as ε is increased the truncated series for velocities fail to converge to an answer for field points on or near the wall and x being near π.

This difficulty was dealt with by using a numerical differentiation scheme referred to as Richardson extrapolation [13]. By this method the series themselves are not differentiated, rather the function (let us say, Ψ) is evaluated on a sequence of

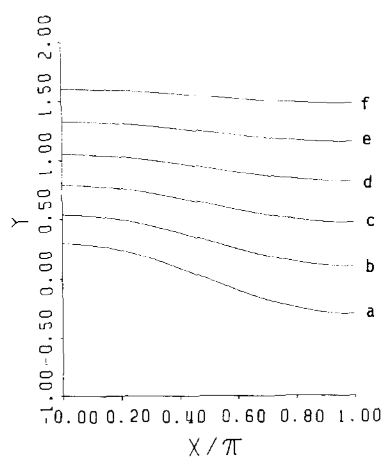

FIG. 2. Streamlines of the flow for $\varepsilon = 0.3$. The contours shown are for Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

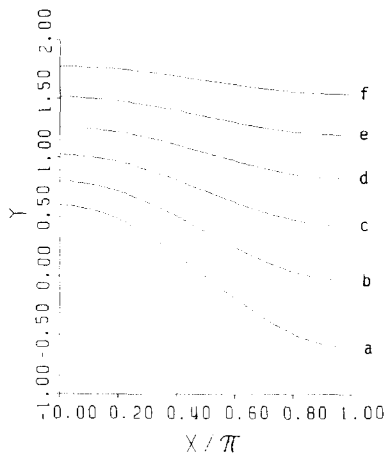

FIG. 3. Streamlines of the flow for $\varepsilon = 0.6$. The contours shown are for Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

points and the derivative is extrapolated by iteration. This proved to be most successful as gauged by the rapid convergence of values in the iteration matrix.

The ability of this scheme to produce accurate estimates of the field velocities can be judged by comparison with the results from [6]. Three points on the wall were chosen for comparison: $(0, \varepsilon)$, $(\pi/2, 0)$, and $(\pi, -\varepsilon)$. Wall points were chosen because their velocities are the most difficult to evaluate. Due to the exponential dependence on y of the terms in the trial solution this is especially so in the region where the wall lies below $y = 0$. The comparison can be found in Table II and as can be seen the present technique produces answers that are correct to within four percent of the actual answers.

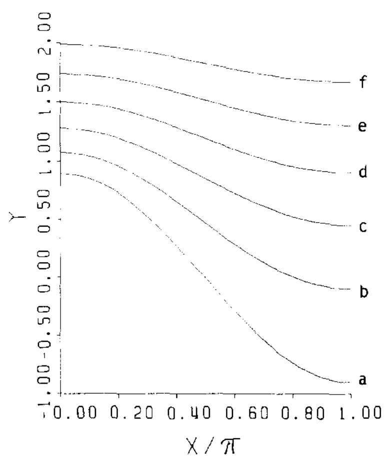

FIG. 4. Streamlines of the flow for $\varepsilon = 0.9$. The contours shown are for Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

TABLE II

Comparison of Wall Speeds Produced by Direct Variation and Euler Transformed Extended Perturbation Series

Direct variation	Euler extended series
$\varepsilon = 0.1$	
$Q(0, \varepsilon) = 1.099759254$	1.099759254
$Q(\pi/2, 0) = 0.995028967$	0.995028967
$Q(\pi, -\varepsilon) = 0.900257283$	0.900257282
$\varepsilon = 0.3$	
$Q(0, \varepsilon) = 1.294111792$	1.294111850
$Q(\pi/2, 0) = 0.957225365$	0.957225440
$Q(\pi, -\varepsilon) = 0.707147763$	0.707147813
$\varepsilon = 0.6$	
$Q(0, \varepsilon) = 1.560852206$	1.561042966
$Q(\pi/2, 0) = 0.850578434$	0.850428223
$Q(\pi, -\varepsilon) = 0.455476411$	0.455612276
$\varepsilon = 0.9$	
$Q(0, \varepsilon) = 1.771863368$	1.793803497
$Q(\pi/2, 0) = 0.722255248$	0.719762966
$Q(\pi, -\varepsilon) = 0.279485804$	0.270335076

Figures 5–8 show the fluid speed Q for the streamline values used in the first set of plots. It can be seen in Fig. 8 that for $\varepsilon = 0.9$ the computed velocity on the wall near $x = \pi$ is not monotonic as we know it should be. The periodic wiggle that appears in the solution is characteristic of the shortest period Fourier component used in the approximating sequence.

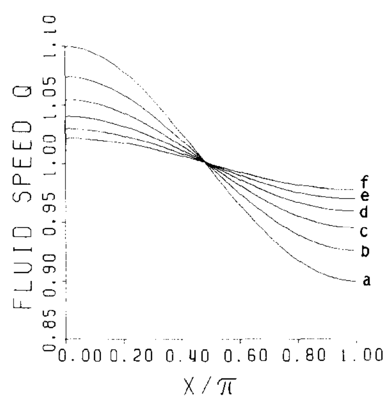

FIG. 5. Fluid speeds for $\varepsilon = 0.1$ along the streamlines Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

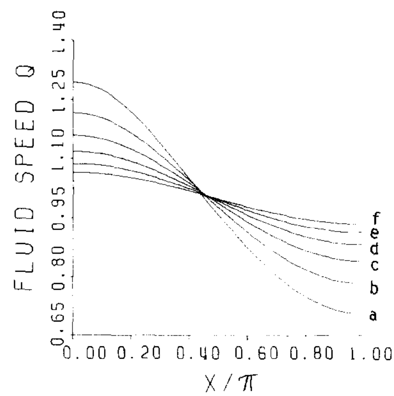

FIG. 6. Fluid speeds for $\varepsilon = 0.3$ along the streamlines Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

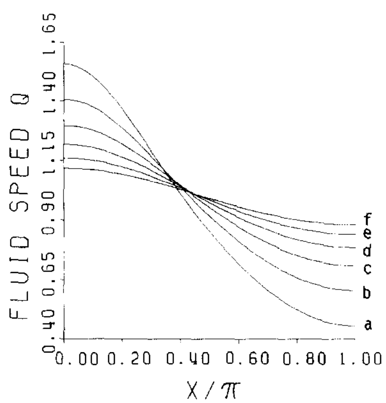

FIG. 7. Fluid speeds for $\varepsilon = 0.6$ along the streamlines Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

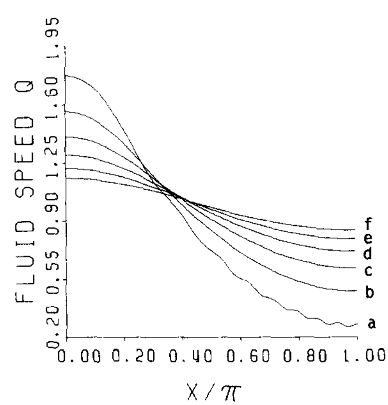

FIG. 8. Fluid speeds for $\varepsilon = 0.9$ along the streamlines Ψ equalling: (a) 0, (b) 0.3, (c) 0.6, (d) 0.9, (e) 1.2, and (f) 1.5.

4. Conclusions

A numerical solution, using the direct variational method, was found for the problem of plane potential flow past a sinusoidal wall of finite height. The advantage of this solution over that of the computer-extended series solution previously published is that it requires relatively few coefficients to be computed and yet still is capable of producing accurate streamlines and field velocities for wall heights of 0.9 or less. Having determined the coefficients of Φ and Ψ, and using the straightforward techniques described here, it is a simple matter to find the streamlines or the fluid speed at any point in the flow field.

Though the direct variational method leads to ill-conditioned matrices for large wall heights this is not a true impediment in this particular problem. The adverse pressure gradients that can be deduced from the fluid velocity plots indicate that even for $\varepsilon = 0.9$ any amount of viscosity will probably cause the flow to separate on the wall. Thus the physical approximations truly fail before the numerical method does.

The advantage of this technique over the finite element approach is that it is not necessary to predetermine which points in the flow-field are of interest. Therefore if this datum is being used as input to some other program, as it currently is, it is more easily attained for any random field point. Also because this method is straightforward it requires less programming than the finite element scheme.

The direct variational method has less advantage over a boundary integral method in solving this problem. Boundary methods for solving Laplace's equation can cope fairly easily with the geometry of the wall and the semi-infinite domain. They are certainly much more flexible in the sense that they can be used in geometries where the use of the direct variational method would be futile. But I do believe that in this case the actual implementation of the direct variational method is more straight forward. This is strictly fortuitous: the geometry leads to a series solution that is a term-by-term solution of the field equation and far-field boundary condition. Too, the solution here is a truncated series that is easy to manipulate and use, and perhaps more easily interpreted in a physical way than the matrix of numbers that would come from a boundary element method. This author is unaware of any published work using this method to solve this problem.

APPENDIX A: Derivation of the Governing Equations from the Variational Principle

Starting with Eq. (6), and recalling Eqs. (1) and (2)

$$u = 1 + \Phi_x \qquad \text{and} \qquad v = \Phi_y \tag{A1}$$

the expression for J becomes

$$J = \int_{-\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} [\Phi_x + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2)] \, dy \, dx. \tag{A2}$$

Using Leibniz's rule and the periodicity of the flow it may be deduced that

$$\int_{-\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} \Phi_x \, dy \, dx = -\int_{-\pi}^{\pi} \Phi(x, \varepsilon \cos x) \, \varepsilon \sin x \, dx \tag{A3}$$

and therefore

$$J = \int_{-\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} \tfrac{1}{2} [\Phi_x^2 + \Phi_y^2] \, dy \, dx - \int_{-\pi}^{\pi} \Phi(x, \varepsilon \cos x) \, \varepsilon \sin x \, dx. \tag{A4}$$

Taking the first variation of J by means of the commonly used "δ" operator one finds

$$\delta J = \int_{\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} \left[\frac{\partial \Phi}{\partial x} \frac{\partial \delta \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \delta \Phi}{\partial y} \right] dy \, dx - \int_{-\pi}^{\pi} \delta \Phi \sin x \Big|_{y = \varepsilon \cos x} dx \tag{A5}$$

Integrating by parts and again making use of Liebniz's rule and the periodicity of the flow reveals

$$\begin{aligned} \delta J = &-\int_{-\pi}^{\pi} \int_{\varepsilon \cos x}^{\infty} [\Phi_{xx} + \Phi_{yy}] \, \delta \Phi \, dy \, dx \\ &-\int_{-\pi}^{\pi} [\Phi_y + \Phi_x \varepsilon \sin x + \varepsilon \sin x] \, \delta \Phi \Big|_{y = \varepsilon \cos x} dx \\ &+\int_{-\pi}^{\pi} \Phi_y \, \delta \Phi \Big|_{y = \varepsilon \cos x} dx \end{aligned} \tag{A6}$$

For the variation of J to be zero, either $\delta \Phi$ is zero or

$$\Phi_{xx} + \Phi_{yy} = 0, \qquad -\pi < x < \pi, \ \varepsilon \cos x < y < \infty, \tag{A7}$$

$$\Phi_y + \Phi_x \varepsilon \sin x + \varepsilon \sin x = 0 \qquad y = \varepsilon \cos x, \ -\pi \leqq x \leqq \pi, \tag{A8}$$

$$\Phi_y \to 0 \qquad y \to \infty, \ -\pi \leqq x \leqq \pi. \tag{A9}$$

Due to the periodicity of the flow these are the governing equations as stated in Eqs. (3)–(5).

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